

Branch: All courses

Sem.: - I

Subject with Subject Code: Engineering Mathematics -I (MATH101) Marks: 60

Date: 11/12/2017

Time:- 3 Hr.

## MODEL SOLUTION WITH OF SCHEME MARKING

Q.N.	Sub. Q.N.		Ma- rks
1.	a)	<p>From given system of linear equations</p> $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : 1 \\ 1 & 2 & 4 & : \lambda \\ 1 & 4 & 10 & : \lambda^2 \end{bmatrix}$ <p>Apply <math>R_2 - R_1</math>, <math>R_3 - R_1</math>, and then <math>R_3 - 3R_1</math>,</p> $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : 1 \\ 0 & 1 & 3 & : \lambda - 1 \\ 0 & 0 & 0 & : \lambda^2 - 3\lambda + 2 \end{bmatrix} \dots\dots\dots(A)$ <p>Here <math>\rho(A) = 2</math>. The system will be consistent if <math>\rho(A) = \rho(A:B) = 2</math>. Therefore <math>\lambda^2 - 3\lambda + 2 = 0</math> i.e. <math>\lambda = 1, 2</math></p> <p>Case I: If <math>\lambda = 1</math> then the system reduces as</p> $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : 1 \\ 0 & 1 & 3 & : 0 \\ 0 & 0 & 0 & : 0 \end{bmatrix}$ <p><math>\rho(A) = \rho(A:B) = 2 &lt; n \therefore</math> The system has infinite solutions.</p> <p><math>R_2</math> shows <math>y + 3z = 0</math>, let <math>z = t \therefore y = -3t</math>  <math>R_1</math> shows <math>x + (-3t) + t = 1 \therefore x = 1 + 2t</math>  Hence for <math>\lambda = 1</math> the solution is <math>x = 1 + 2t</math>, <math>y = -3t</math>, <math>z = t</math>.</p> <p>Case II: If <math>\lambda = 2</math> then the system reduces as</p> $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : 1 \\ 0 & 1 & 3 & : 1 \\ 0 & 0 & 0 & : 0 \end{bmatrix}$ <p><math>\rho(A) = \rho(A:B) = 2 &lt; n \therefore</math> the system has infinite solutions.</p> <p><math>R_2</math> shows <math>y + 3z = 1</math>, let <math>z = t \therefore y = 1 - 3t</math>  <math>R_1</math> shows <math>x + (1 - 3t) + t = 1 \therefore x = 2t</math>  Hence for <math>\lambda = 2</math> the solution is <math>x = 2t</math>, <math>y = 1 - 3t</math>, <math>z = t</math>.</p>	01 01 01 01 01 01 01 01
	b)	<p>Step I- :To find the eigen values -</p> <p>We know the characteristic equation of matrix A is <math> A - \lambda I  = 0</math></p> $\lambda^3 - S_1\lambda^2 + S_2\lambda -  A  = 0$ $\lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0$	

solving we get  $\lambda = -3, 3, 9$

Step-II: To find corresponding eigen vectors-

Case-I: For  $\lambda = 3$  the matrix equation

$$[A - \lambda I][X] = [0] \text{ reduces}$$

$$\begin{bmatrix} -2 & 0 & -4 \\ 0 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By  $R_1$  and  $R_2$  we have

$$2x + 0y + 4z = 0$$

$$x + 2y + 4z = 0$$

$$\text{By Crammers rule } \frac{x}{2} = \frac{-y}{-2} = \frac{z}{-1} \text{ this gives } X_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Case-II: For  $\lambda = -3$  the matrix equation  $[A - \lambda I][X] = [0]$  reduces

$$\begin{bmatrix} 4 & 0 & -4 \\ 0 & 8 & 4 \\ -4 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By  $R_1$  and  $R_2$  we have

$$4x + 0y - 4z = 0 \text{ and } 0x + 8y + 4z = 0$$

$$\text{By Crammers rule } \frac{x}{2} = \frac{-y}{1} = \frac{z}{2} \text{ this gives } X_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Case-III: For  $\lambda = 9$  the matrix equation  $[A - \lambda I][X] = [0]$  reduces

$$\begin{bmatrix} -8 & 0 & 4 \\ 0 & 4 & 4 \\ -4 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By  $R_1$  and  $R_2$  we have

$$-8x + 0y + 4z = 0$$

$$0x + 4y + 4z = 0$$

$$\text{By Crammers rule } \frac{x}{1} = \frac{-y}{2} = \frac{z}{2} \text{ this gives } X_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

2. a) Given  $y = \sin px + \cos px$   
Diff. w.r.t  $x, n$  times

$$y_n = p^n \left[ \sin \left( px + \frac{n\pi}{2} \right) + \cos \left( px + \frac{n\pi}{2} \right) \right]$$

$$y_n = p^n \left[ \left\{ \sin \left( px + \frac{n\pi}{2} \right) + \cos \left( px + \frac{n\pi}{2} \right) \right\}^2 \right]^{\frac{1}{2}}$$

$$y_n = p^n \left[ \left\{ 1 + \sin \left( 2px + \frac{2n\pi}{2} \right) \right\} \right]^{\frac{1}{2}}$$

$$y_n = p^n [1 + (-1)^n \sin 2px]^{\frac{1}{2}}$$

01

01

01

01

01

01

01

01

01

01

01

	B	<p>Given <math>y = e^{a \cos^{-1} x}</math>          Diff. w.r.t <math>x</math></p> $y_1 = e^{a \cos^{-1} x} \left\{ \frac{-a}{\sqrt{1-x^2}} \right\}$ $(1-x^2)y_1^2 = a^2 y^2$ <p>Again diff, w.r.t <math>x</math></p> $(1-x^2)y_2 - xy_1 = a^2 y$ <p>Now diff. <math>n</math> times w.r.t <math>x</math> by Leibnitz's Rule  <math>D^n[(1-x^2)y_2] - D^n(xy_1) = D^n(a^2 y)</math></p> $(1-x^2)y_{n+2} - (2n+1)xy_1 - (n^2 + a^2)y_n = 0$	(i) 01 (ii) 01 (iii) 01 (iv) 01
	c)	<p>We know that Taylor's series expansion state that</p> $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots \quad (A)$ <p>Here <math>a = \frac{\pi}{3}</math></p> $f(x) = \ln(\cos x) \quad \therefore f(a) = \ln \frac{1}{2}$ $f'(x) = -\tan x \quad \therefore f'(a) = -\sqrt{3}$ $f''(x) = -\sec^2 x \quad \therefore f''(a) = -4$ $f'''(x) = -2\sec^2 x \tan x \quad \therefore f'''(a) = -8\sqrt{3}$ <p>Put all above values in (A)</p> $\ln(\cos x) = \ln \frac{1}{2} - \sqrt{3} \left( x - \frac{\pi}{3} \right) - 2 \left( x - \frac{\pi}{3} \right)^2 - \frac{4}{3}\sqrt{3} \left( x - \frac{\pi}{3} \right)^3 + \dots$	01 01 01 01
3.	a)	<p>Given <math>x^x y^y z^z = c</math>          Taking <math>\ln</math> of both sides</p> $x \ln x + y \ln y + z \ln z = \ln c$ <p>diff w.r.t. <math>y</math> (keeping <math>x</math> constant)</p> $0 + \left\{ y \frac{1}{y} + \ln y \cdot 1 \right\} + \left\{ z \frac{1}{z} + \ln z \cdot 1 \right\} \frac{\partial z}{\partial y} = 0$ $\Rightarrow \frac{\partial z}{\partial y} = \frac{-(1 + \ln y)}{(1 + \ln z)} \quad (A)$ <p>Similarly,</p> $\Rightarrow \frac{\partial z}{\partial x} = \frac{-(1 + \ln x)}{(1 + \ln z)} \quad (B)$ <p>At point <math>x = y = z</math>, both <math>\Rightarrow \frac{\partial z}{\partial y} = -1 = \frac{\partial z}{\partial x}</math></p> <p>Now, diff (A) w.r.t 'x'</p>	01 01 01

$$\frac{\partial^2 z}{\partial x \partial y} = -(1 + \ln y) \frac{\partial}{\partial x} \left\{ \frac{1}{(1 + \ln z)} \right\}$$

$$= -(1 + \ln y) \left\{ \frac{-1}{(1 + \ln z)^2} * \frac{1}{z} * \frac{\partial z}{\partial x} \right\}$$

01

At  $x = y = z$ 

$$\frac{\partial^2 z}{\partial x \partial y} = (1 + \ln x) \left\{ \frac{1}{(1 + \ln x)^2} * \frac{1}{x} * (-1) \right\}$$

$$= \frac{-1}{x(\ln e + \ln x)}$$

$$= -\{x \ln ex\}^{-1}$$

01

b)  $u = \tan^{-1} \left( \frac{x^3 + y^3}{x - y} \right)$

$$\tan u = \left( \frac{x^3 + y^3}{x - y} \right) = z, \quad \text{say}$$

01

$\therefore z = \left( \frac{x^3 + y^3}{x - y} \right)$  is homogenous function of degree  $n = 2$

Also,  $z = \tan u \quad \therefore f(u) = \tan u \quad \therefore f'(u) = \sec^2 u$

$\therefore$  By Euler's deduction I

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$$

$$= 2 \tan u \cos^2 u$$

$$= 2 \sin u \cos u = \sin 2u = g(u), \quad \text{say}$$

01

By Euler's deduction II

$$x^2 U_{xx} + 2xy U_{xy} + y^2 U_{yy} = g(u)[g'(u) - 1]$$

01

$$= \sin 2u [2 \cos 2u - 1]$$

$$= 2 \sin 2u \cos 2u - \sin 2u$$

$$= \sin 4u - \sin 2u$$

01

	$= 2 \cos 3u \cdot \sin u$	
C	from given equations, we get  $z \rightarrow x, y \rightarrow u, v$ As $x^2 = au + bv$ $\therefore 2x \frac{\partial x}{\partial u} = a$ $y^2 = au - bv$ $\therefore 2y \frac{\partial y}{\partial u} = a$ $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$ $= \frac{\partial z}{\partial x} \cdot \left(\frac{a}{2x}\right) + \frac{\partial z}{\partial y} \cdot \left(\frac{a}{2y}\right)$ $2u \cdot \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \left(\frac{au}{x}\right) + \frac{\partial z}{\partial y} \cdot \left(\frac{au}{y}\right) \quad (A)$ $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \left(\frac{b}{2x}\right) + \frac{\partial z}{\partial y} \cdot \left(\frac{-b}{2y}\right)$ $2v \cdot \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \left(\frac{bv}{x}\right) + \frac{\partial z}{\partial y} \cdot \left(\frac{-bv}{y}\right) \quad (B)$ Adding (A) and (B) gives, $2 \left[ u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} \right] = \frac{\partial z}{\partial x} \cdot \left(\frac{au + bv}{x}\right) + \frac{\partial z}{\partial y} \cdot \left(\frac{au - bv}{y}\right)$ $= x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$	01 01 01
d)	Here, $u \rightarrow x, y \rightarrow t$ $\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$ $= \left(\cos\left(\frac{x}{y}\right)\right) \cdot \frac{1}{y} (e^t) + \left(\cos\left(\frac{x}{y}\right)\right) \cdot \frac{(-x)}{y^2} (2t)$	01 01

$$= \cos\left(\frac{x}{y}\right) \left[ \frac{1}{y}(e^t) - \frac{x}{y^2}(2t) \right]$$

01

$$\cos\left(\frac{e^t}{t^2}\right) \left[ \frac{1}{t^2}(e^t) - \frac{2}{t^3} \right] e^t$$

01

4 a) From given equations,  
 $u = \frac{yz}{x}, \quad v = \frac{zx}{y}, \quad w = \frac{xy}{z}$

$$\therefore u, v, w \rightarrow x, y, z$$

To find J,

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

01

$$= \frac{1}{x^2 y^2 z^2} \begin{vmatrix} -yz & zx & xy \\ yz & -zx & xy \\ yz & zx & -xy \end{vmatrix}$$

$$= \frac{yz.zx.xy}{x^2 y^2 z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4 \quad (\text{A})$$

01

To find  $J'$ :

$$f_1 = \frac{yz}{x} - u = 0, \quad f_2 = \frac{zx}{y} - v = 0, \quad f_3 = \frac{xy}{z} - w = 0$$

$$J' = \frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)/\partial(u, v, w)}{\partial(f_1, f_2, f_3)/\partial(x, y, z)}$$

01

$$= (-1)^3 \cdot \frac{\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}}$$

$$= \frac{1}{4}$$

(B)

		from Equation (A) and (B)	
		$J \cdot J' = 1$	01
b)	From sine rule,		
	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$		
	$\Rightarrow a = 2R \sin A \quad \Rightarrow da = 2R \cos A \, dA$		01
	$\Rightarrow \frac{da}{\cos A} = 2R \, dA \quad (i)$		
	Similarly,		
	$\Rightarrow b = 2R \sin B \quad \Rightarrow db = 2R \cos B \, dB$		01
	$\Rightarrow \frac{db}{\cos B} = 2R \, dB \quad (ii)$		
	And,		
	$\Rightarrow c = 2R \sin C \quad \Rightarrow dc = 2R \cos C \, dC$		01
	$\Rightarrow \frac{dc}{\cos C} = 2R \, dC \quad (iii)$		
	Adding Above Results,		
	$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R \, d(A + B + C) = 0$		01
c)	Let the dimension of the box be $x, y, z$		
	$S = xy + 2yz + 2zx \quad (i)$		
	$V = xyz \quad (ii)$		
	the material required is least if its surface area is minimum.		
	Construct new function		
	$F = S + \lambda V$		01
	$F = (xy + 2yz + 2zx) + \lambda(xyz)$		
	Differentiate $F$ with respect to $x, y & z$ respectively, and equate it to 0		
	$\frac{\partial F}{\partial x} = (y + 2z) + \lambda(yz) = 0 \quad (iii)$		01
	$\frac{\partial F}{\partial y} = (x + 2z) + \lambda(xz) = 0 \quad (iv)$		

$$\frac{\partial F}{\partial z} = (2y + 2x) + \lambda(xy) = 0 \quad (v)$$

$x$ (iii),  $y$  (iv),  $z$  (v) and equating all get  
 $xy + 2zx = xy + 2zy = 2yz + 2xz = -\lambda xyz$

01

$$\Rightarrow 2xz = 2yz, \quad xy = 2zx$$

$$\Rightarrow x = y, \quad \text{and } y = 2z$$

$$\therefore x = y = 2z$$

Put in equation (ii)

$$(2z) \cdot (2z) \cdot z = 32$$

$$\therefore z^3 = 8, \quad \Rightarrow z = 2$$

The required dimension are,

$$x = 4, y = 4 \text{ and } z = 2$$

01

d)  $f(x, y) = x^y \Rightarrow f(1, 1) = 1$

01

$$\Rightarrow f_y = x^y \ln x \Rightarrow f_y(1, 1) = 0$$

$$\Rightarrow f_x = yx^{y-1} \Rightarrow f_x(1, 1) = 1$$

$$f_{xx} = y(y-1)x^{y-2}, \quad \Rightarrow f_{xx}(1, 1) = 0$$

$$f_{xy} = y \cdot x^{(y-1)} \cdot \ln x + x^{(y-1)}, \Rightarrow f_{xy}(1, 1) = 1$$

$$f_{yy} = \ln x (x^y \cdot \ln x), \quad \Rightarrow f_{yy}(1, 1) = 0$$

01

Putting all above values in Taylor's theorem

$$\begin{aligned} f(x, y) &= f(a, b) + [(x-1)f_x(a, b) + (y-1)f_y(a, b)] \\ &\quad + \frac{1}{2!} [(x-1)^2 f_{xx}(a, b) + 2(x-1)(y-1)f_{xy}(a, b) \\ &\quad + (y-1)^2 f_{yy}(a, b)] + \dots \dots \dots \end{aligned}$$

01

$$x^y = 1 + [(x-1) + 0]$$

$$+ \frac{1}{2!} [(x-1)^2 \cdot 0 + 2(x-1)(y-1) \cdot 1 + (y-1)^2 \cdot 0]$$

01

$$= 1 + (x-1) + (x-1)(y-1) + \dots \dots \dots$$

5

a)

Given limits are,

$$y = x, \text{ to } y = \pi/2$$

and

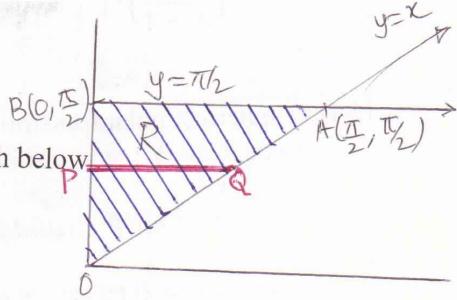
$$x = 0, \text{ to } x = \pi/2$$

which shows the region as sketch below

Change the order of integration

∴ Draw a strip parallel to x-axis

$$\therefore x_P = 0 \text{ to } x_Q = y \\ \text{and } y = 0 \text{ to } y = \pi/2$$



01

$$\therefore I = \int_{y=0}^{\frac{\pi}{2}} \left[ \int_{x=0}^y dx \right] \frac{\cos y}{y} \cdot dy$$

$$= \int_{y=0}^{\frac{\pi}{2}} (x)_0^y \frac{\cos y}{y} \cdot dy$$

$$= \int_{y=0}^{\frac{\pi}{2}} \cos y \cdot dy$$

$$= (\sin y)_0^{\frac{\pi}{2}}$$

$$= 1$$

01

01

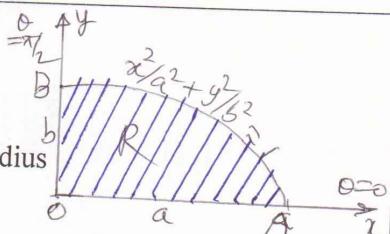
01

b)

Use Elliptical polar transformation

$$\begin{cases} x = a \cos \theta \\ y = b \sin \theta \end{cases} \Rightarrow dx dy = abr dr d\theta$$

∴ ellipse get reduce into circle of unit radius

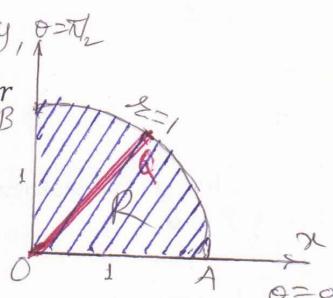


01

$$\therefore I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{x=0}^y (abr \cos \theta)(abr \sin \theta) dx \cdot (r^2)^{\frac{n}{2}} \cdot abr dr d\theta$$

$$= a^2 b^2 \int_{\theta=0}^{\frac{\pi}{2}} \sin \theta \cdot \cos \theta d\theta \cdot \int_{r=0}^1 r^{n+3} dr$$

$$= a^2 b^2 \left( \frac{\sin^2 \theta}{2} \right)_0^{\frac{\pi}{2}} \cdot \left( \frac{r^{n+4}}{n+4} \right)_0^1$$



01

01

		$= \frac{a^2 b^2}{2} \cdot \left( \frac{1}{n+4} \right)$	01
c)	Using spherical polar transformation $x = r \cos \theta \sin \phi ; y = r \sin \theta \sin \phi ; z = r \cos \theta$ $\therefore dx dy dz = r^2 \sin \theta dr d\theta d\phi$		01
	As limits are $x = 0$ to $\infty$ ; $y = 0$ to $\infty$ ; $z = 0$ to $\infty$ implies, the volume is in positive octant of sphere of $\infty$ radius, whose standard limits are, $\phi = 0$ to $\frac{\pi}{2}$ ; $\theta = 0$ to $\frac{\pi}{2}$ ; $r = 0$ to $\infty$ ;		01
	$\therefore I = \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} \frac{r^2 \sin \theta}{(1+r^2)^2} dr d\theta d\phi$		01
	$= \int_{\phi=0}^{\frac{\pi}{2}} d\phi \int_{\theta=0}^{\frac{\pi}{2}} \sin \theta d\theta \int_{r=0}^{\infty} \frac{r^2}{(1+r^2)^2} dr$		01
	$= (\phi)_{0}^{\frac{\pi}{2}} \cdot (-\cos \theta)_{0}^{\frac{\pi}{2}} \cdot \int_{t=0}^{\frac{\pi}{2}} \frac{\tan^2 t}{\sec^4 t} \cdot \sec^2 t dt$		01
	$= \left(\frac{\pi}{2} - 0\right) \cdot (1 - 0) \cdot \int_{t=0}^{\frac{\pi}{2}} \sin^2 t dt$		01
	$= \frac{\pi}{2} \cdot 1 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$		01
	$= \frac{\pi^2}{8}$		01
d)	Since the loop is symmetrical about x-axis implies, $\bar{y} = 0$ $\bar{x} = \frac{\iint x dx dy}{\iint dx dy}$		

$$= \frac{\int \int r \cos \theta \cdot r dr d\theta}{\int \int r dr d\theta}$$

01

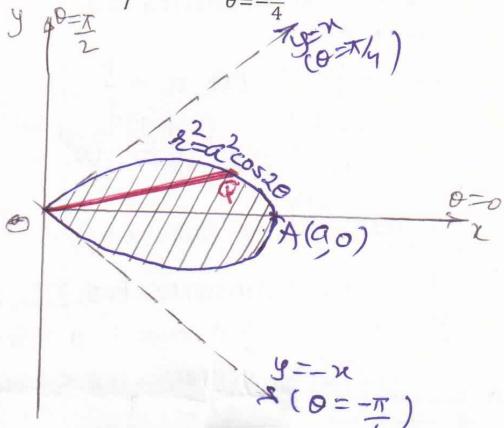
$$= \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos \theta \left( \frac{r^3}{3} \right)_0^{a\sqrt{\cos \theta}} d\theta \quad \left/ \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} \left( \frac{r^2}{2} \right)_0^{a\sqrt{\cos \theta}} d\theta \right.$$

01

$$= \frac{2a^3}{3} \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 - 2 \sin^2 \theta)^{\frac{3}{2}} \cdot \cos \theta d\theta \quad \left/ \frac{2a^2}{2} \cdot \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta d\theta \right.$$

01

$$= \frac{2a^3}{3\sqrt{2}} \int_0^{\frac{\pi}{2}} \cos^4 t dt \quad \left/ \frac{a^2}{2} \right.$$



$$= \frac{\pi a^3}{8\sqrt{2}} / \frac{a^2}{2}$$

$$= \frac{\pi a}{4\sqrt{2}}$$

$$\therefore \text{C.G. is } \left( \frac{\pi a}{4\sqrt{2}}, 0 \right)$$

01

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{n^2}{2^n} + \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (A)$$

01

$$\text{Let } U_n = \frac{n^2}{2^n} \text{ and } V_n = \frac{1}{n^2}$$

$$U_{n+1} = \frac{(n+1)^2}{2^{(n+1)}}$$

$$\lim_{n \rightarrow \infty} \frac{U_n}{U_{n+1}} = \lim_{n \rightarrow \infty} 2 \left( \frac{n}{n+1} \right)^2 = \lim_{n \rightarrow \infty} 2 \left( \frac{1}{1 + \frac{1}{n}} \right)^2 = 2 > 1$$

∴ by D'Alembert's Ratio test  $\sum_{n=1}^{\infty} U_n$  is converges ..... (ii)

01

$\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$  is converges by p-series test.

01

As the sum of the two convergent series is convergent.  
 $\therefore$  The given infinite series is converges.

01

b)

$$u_n = \frac{(n+1)^n}{n^{n+1}} x^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right) \frac{1}{n^{1/n}} x$$

$$= \lim_{n \rightarrow \infty} \left( \frac{1 + \frac{1}{n}}{1} \right) \lim_{n \rightarrow \infty} \left( \frac{1}{n^{1/n}} \right) x = x$$

01

Hence by Cauchy's root test the series is converges if  $x < 1$  diverges if  $x > 1$  and test fails if  $x = 1$

01

For  $x = 1$ ,

$$u_n = \frac{(n+1)^n}{n^{n+1}} . \text{ Let } v_n = \frac{1}{n}$$

$$\therefore \lim_{n \rightarrow \infty} \left| \frac{u_n}{v_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^n}{n^{n+1}} (n)$$

$$= \lim_{n \rightarrow \infty} \left( \frac{1 + \frac{1}{n}}{1} \right)^n = e \text{ finite } (\neq 0)$$

01

Hence by Comparison test both  $\sum_{n=1}^{\infty} u_n$  and  $\sum_{n=1}^{\infty} v_n$  behaves alike.  
 $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$  diverges by p-series test.

01

Hence  $\sum_{n=1}^{\infty} u_n$  converges if  $x < 1$  and diverges if  $x \geq 1$

c)

Given series is an alternating series

$$\sum_{n=2}^{\infty} u_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$$

$$u_n = \frac{1}{2(\ln 2)^2} - \frac{1}{3(\ln 3)^2} + \frac{1}{4(\ln 4)^2} - \frac{1}{5(\ln 5)^2} + \dots$$

01

is alternating series with each term is less than preceding term  
 $\text{Also } \lim_{n \rightarrow \infty} |u_n| = 0$

01

Both the conditions of Leibnitz's test satisfied.

Now

$$\sum_{n=2}^{\infty} |u_n| = \int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \int \frac{1}{t^2} dt$$

01

$$= \left[ -\frac{1}{\ln x} \right] = \frac{1}{\ln 2} \text{ is finite}$$

01

Hence by Integral test given series  $\sum_{n=2}^{\infty} u_n$  is convergent.