Solution and Marking Scheme for subject code MTEE103

Q.No.1 Attempt any **one** of the following 08 Marks

a) Obtain the state model for the system described by

T(s) = =

Using phase variable form.

Solution: The differential equation corresponding to the given transfer function is obtained by cross-multiplying and taking the inverse Laplace transform. So, we have

$$\dddot{y} + 6\ddot{y} + 10\dot{y} + 5y = u$$

Since the derivatives of the input are not present in the differential equation, phase variables can be selected as the state variables. Therefore,

$x_1 = y$ i.e. $y = x_1$

$x_2 = \dot{y} = \dot{x}_1$ $\quad \dot{x}_1 = x_2$

$x_3 = \ddot{y} = \dot{x}_2$ $\quad \dot{x}_2 = x_3$

$\dddot{y} = -6\ddot{y} - 10\dot{y} - 5y + u$ $\quad \dot{x}_3 = -5x_1 - 10x_2 - 6x_3 + u$

Therefore, the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -10 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

08 Marks

b) Explain necessary and sufficient condition for Arbitrary Pole Placement.

Necessary and Sufficient Condition for Arbitrary Pole Placement We shall now prove that a necessary and sufficient condition for arbitrary pole placement is that the system be completely state controllable. We shall first derive the necessary condition. We begin by proving that if the system is not completely state controllable, then there are eigenvalues of matrix $\mathbf{A} - \mathbf{BK}$ that cannot be controlled by state feedback.

Suppose that the system of Equation (10–1) is not completely state controllable. Then the rank of the controllability matrix is less than n, or

$$\operatorname{rank}\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right] = q < n$$

This means that there are q linearly independent column vectors in the controllability matrix. Let us define such q linearly independent column vectors as $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_q$. Also, let us choose $n - q$ additional n-vectors $\mathbf{v}_{q+1}, \mathbf{v}_{q+2}, \ldots, \mathbf{v}_n$ such that

$$\mathbf{P} = \left[\mathbf{f}_1 \;\vdots\; \mathbf{f}_2 \;\vdots\; \cdots \;\vdots\; \mathbf{f}_q \;\vdots\; \mathbf{v}_{q+1} \;\vdots\; \mathbf{v}_{q+2} \;\vdots\; \cdots \;\vdots\; \mathbf{v}_n\right]$$

is of rank n. Then it can be shown that

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right], \qquad \hat{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B} = \left[\begin{array}{c} \mathbf{B}_{11} \\ \hline \mathbf{0} \end{array}\right]$$

(See Problem **A–10–1** for the derivation of these equations.) Now define

$$\hat{\mathbf{K}} = \mathbf{KP} = \left[\mathbf{k}_1 \;\vdots\; \mathbf{k}_2\right]$$

Then we have

$$\begin{aligned}
|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| &= |\mathbf{P}^{-1}(s\mathbf{I} - \mathbf{A} + \mathbf{BK})\mathbf{P}| \\
&= |s\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP} + \mathbf{P}^{-1}\mathbf{BKP}| \\
&= |s\mathbf{I} - \hat{\mathbf{A}} + \hat{\mathbf{B}}\hat{\mathbf{K}}| \\
&= \left| s\mathbf{I} - \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right] + \left[\begin{array}{c} \mathbf{B}_{11} \\ \hline \mathbf{0} \end{array}\right]\left[\mathbf{k}_1 \;\vdots\; \mathbf{k}_2\right] \right| \\
&= \begin{vmatrix} s\mathbf{I}_q - \mathbf{A}_{11} + \mathbf{B}_{11}\mathbf{k}_1 & -\mathbf{A}_{12} + \mathbf{B}_{11}\mathbf{k}_2 \\ \mathbf{0} & s\mathbf{I}_{n-q} - \mathbf{A}_{22} \end{vmatrix} \\
&= |s\mathbf{I}_q - \mathbf{A}_{11} + \mathbf{B}_{11}\mathbf{k}_1| \cdot |s\mathbf{I}_{n-q} - \mathbf{A}_{22}| = 0
\end{aligned}$$

where $\mathbf{I}_q$ is a q-dimensional identity matrix and $\mathbf{I}_{n-q}$ is an $(n - q)$-dimensional identity matrix.

Notice that the eigenvalues of $\mathbf{A}_{22}$ do not depend on $\mathbf{K}$. Thus, if the system is not completely state controllable, then there are eigenvalues of matrix $\mathbf{A}$ that cannot be arbitrarily placed. Therefore, to place the eigenvalues of matrix $\mathbf{A} - \mathbf{BK}$ arbitrarily, the system must be completely state controllable (necessary condition).

Next we shall prove a sufficient condition: that is, if the system is completely state controllable, then all eigenvalues of matrix $\mathbf{A}$ can be arbitrarily placed.

In proving a sufficient condition, it is convenient to transform the state equation given by Equation (10–1) into the controllable canonical form.

Define a transformation matrix $\mathbf{T}$ by

$$\mathbf{T} = \mathbf{MW} \tag{10-4}$$

where $\mathbf{M}$ is the controllability matrix

$$\mathbf{M} = \left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right] \tag{10-5}$$

and

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{10-6}$$

where the a_i's are coefficients of the characteristic polynomial

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

Define a new state vector $\hat{\mathbf{x}}$ by

$$\mathbf{x} = \mathbf{T}\hat{\mathbf{x}}$$

If the rank of the controllability matrix $\mathbf{M}$ is n (meaning that the system is completely state controllable), then the inverse of matrix $\mathbf{T}$ exists, and Equation (10–1) can be modified to

$$\dot{\hat{\mathbf{x}}} = \mathbf{T}^{-1}\mathbf{AT}\hat{x} + \mathbf{T}^{-1}\mathbf{B}u \tag{10-7}$$

where

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \tag{10-8}$$

$$\mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{10–9}$$

[See Problems **A–10–2** and **A–10–3** for the derivation of Equations (10–8) and (10–9).] Equation (10–7) is in the controllable canonical form. Thus, given a state equation, Equation (10–1), it can be transformed into the controllable canonical form if the system is completely state controllable and if we transform the state vector $\mathbf{x}$ into state vector $\hat{\mathbf{x}}$ by use of the transformation matrix $\mathbf{T}$ given by Equation (10–4).

Let us choose a set of the desired eigenvalues as $\mu_1, \mu_2, \ldots, \mu_n$. Then the desired characteristic equation becomes

$$(s - \mu_1)(s - \mu_2)\cdots(s - \mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n = 0 \tag{10–10}$$

Let us write

$$\mathbf{KT} = \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{bmatrix} \tag{10–11}$$

When $u = -\mathbf{KT}\hat{\mathbf{x}}$ is used to control the system given by Equation (10–7), the system equation becomes

$$\dot{\hat{\mathbf{x}}} = \mathbf{T}^{-1}\mathbf{AT}\hat{\mathbf{x}} - \mathbf{T}^{-1}\mathbf{BKT}\hat{\mathbf{x}}$$

The characteristic equation is

$$\left|s\mathbf{I} - \mathbf{T}^{-1}\mathbf{AT} + \mathbf{T}^{-1}\mathbf{BKT}\right| = 0$$

This characteristic equation is the same as the characteristic equation for the system, defined by Equation (10–1), when $u = -\mathbf{Kx}$ is used as the control signal. This can be seen as follows: Since

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$$

the characteristic equation for this system is

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \left|\mathbf{T}^{-1}(s\mathbf{I} - \mathbf{A} + \mathbf{BK})\mathbf{T}\right| = \left|s\mathbf{I} - \mathbf{T}^{-1}\mathbf{AT} + \mathbf{T}^{-1}\mathbf{BKT}\right| = 0$$

Now let us simplify the characteristic equation of the system in the controllable canonical form. Referring to Equations (10–8), (10–9), and (10–11), we have

$$
\begin{aligned}
&|s\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T} + \mathbf{T}^{-1}\mathbf{B}\mathbf{K}\mathbf{T}| \\
&= \left| s\mathbf{I} - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{bmatrix} \right| \\
&= \begin{vmatrix} s & -1 & \cdots & 0 \\ 0 & s & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n + \delta_n & a_{n-1} + \delta_{n-1} & \cdots & s + a_1 + \delta_1 \end{vmatrix} \\
&= s^n + (a_1 + \delta_1)s^{n-1} + \cdots + (a_{n-1} + \delta_{n-1})s + (a_n + \delta_n) = 0 \qquad (10\text{–}12)
\end{aligned}
$$

This is the characteristic equation for the system with state feedback. Therefore, it must be equal to Equation (10–10), the desired characteristic equation. By equating the coefficients of like powers of s, we get

$$
\begin{aligned}
a_1 + \delta_1 &= \alpha_1 \\
a_2 + \delta_2 &= \alpha_2 \\
&\ \ \cdot \\
&\ \ \cdot \\
&\ \ \cdot \\
a_n + \delta_n &= \alpha_n
\end{aligned}
$$

Solving the preceding equations for the δ_i's and substituting them into Equation (10–11), we obtain

$$
\begin{aligned}
\mathbf{K} &= \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{bmatrix}\mathbf{T}^{-1} \\
&= \begin{bmatrix} \alpha_n - a_n \vdots \alpha_{n-1} - a_{n-1} \vdots \cdots \vdots \alpha_2 - a_2 \vdots \alpha_1 - a_1 \end{bmatrix}\mathbf{T}^{-1} \qquad (10\text{–}13)
\end{aligned}
$$

Thus, if the system is completely state controllable, all eigenvalues can be arbitrarily placed by choosing matrix **K** according to Equation (10–13) (sufficient condition).

We have thus proved that a necessary and sufficient condition for arbitrary pole placement is that the system be completely state controllable.

It is noted that if the system is not completely state controllable, but is stabilizable, then it is possible to make the entire system stable by placing the closed-loop poles at desired locations for q controllable modes. The remaining $n - q$ uncontrollable modes are stable. So the entire system can be made stable.

Q.No. 2 a) Determine the canonical state model of the system, whose transfer function is 04 Marks

T(s) =

SOLUTION

By partial fraction expansion,

$$\frac{Y(s)}{U(s)} = \frac{2\,(s+5)}{(s+2)\,(s+3)\,(s+4)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$A = \left.\frac{2\,(s+5)}{(s+3)\,(s+4)}\right|_{s=-2} = \frac{2(-2+5)}{(-2+3)\,(-2+4)} = \frac{2\times 3}{1\times 2} = 3$$

$$B = \left.\frac{2\,(s+5)}{(s+2)\,(s+4)}\right|_{s=-3} = \frac{2(-3+5)}{(-3+2)(-3+4)} = \frac{2\times 2}{-1\times 1} = -4$$

$$C = \left.\frac{2\,(s+5)}{(s+2)\,(s+3)}\right|_{s=-4} = \frac{2(-4+5)}{(-4+2)\,(-4+3)} = \frac{2\times 1}{-2\times(-1)} = 1$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{3}{s+2} - \frac{4}{s+3} + \frac{1}{s+4}$$

$$\frac{Y(s)}{U(s)} = \frac{3}{s(1+2/s)} - \frac{4}{s(1+3/s)} + \frac{1}{s(1+4/s)}$$

$$\therefore Y(s) = \left[\frac{\frac{1}{s}}{1+\frac{1}{s}\times 2}\times 3\right]U(s) - \left[\frac{\frac{1}{s}}{1+\frac{1}{s}\times 3}\times 4\right]U(s) + \left[\frac{\frac{1}{s}}{1+\frac{1}{s}\times 4}\right]U(s)$$

b) Explain Gilbert's method of testing controllability. 04 Marks

Gilbert's method of testing controllability

Case(i) : When the system matrix has distinct eigenvalues

In this case the system matrix can be diagonalized and the state model can be converted to canonical form.

Consider the state model of the system,

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

The state model can be converted to canonical form by a transformation, $X = MZ$, where M is the modal matrix and Z is the transformed state variable vector.

The transformed state model is given by

$$\dot{Z} = \wedge Z + \tilde{B}U$$

$$Y = \tilde{C}Z + DU$$

where $\wedge = M^{-1}AM$

$$\tilde{B} = M^{-1}B$$

$$\tilde{C} = CM$$

In this case the necessary and sufficient condition for complete controllability is that, the matrix $\tilde{B}$ must have no rows with all zeros. If any row of the matrix $\tilde{B}$ is zero then the corresponding state variable is uncontrollable.

Case(ii) : When the system matrix has repeated eigenvalues

In this case, the system matrix cannot be diagonalized but can be transformed to Jordan canonical form.

Consider the state model of the system,

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

The state model can be transformed to Jordan canonical form by a transformation $X = MZ$, where M is model matrix and Z is the transformed state variable vector.

The transformed state model is given by,

$$\dot{Z} = J\dot{Z} + \tilde{B}U$$
$$Y = \tilde{C}Z + DU$$

where $J = M^{-1}AM$

$\tilde{B} = M^{-1}B$

$\tilde{C} = CM$

c) Determine the state controllability for the systems represented by the following state equations.

04 Marks

Soln:

Given.

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore \quad AB = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The controllability matrix is given by

$$S = [B \quad AB] = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Since, $S \neq 0$, therefore the system is completely state controllable.

d) Obtain STM for the state model whose A matrix is given below using Cayley-Hamilton theorem.

A =

04 Marks

Solution: (a) The eigenvalues of the system matrix **A** are the roots of the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = \left|\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right| = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)(\lambda - 1) = 0$$

Therefore, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$.

We know that

$$e^{\lambda t} = \alpha_0 + \alpha_1 \lambda$$

Substituting $\lambda = 1$ in the above equation, we have

$$e^t = \alpha_0 + \alpha_1$$

Differentiating the equation $e^{\lambda t} = \alpha_0 + \alpha_1 \lambda$, with respect to λ

$$te^{\lambda t}\Big|_{\lambda = 1} = te^t = \alpha_1$$

Substituting this value of α_1 in the expression for $e^t = \alpha_0 + \alpha_1$, we have

$$\alpha_0 = -\alpha_1 + e^t = -te^t + e^t$$

Therefore, the STM is given by

$$\phi(t) = e^{\mathbf{A}t} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A}$$

$$= \alpha_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{bmatrix} + \begin{bmatrix} \alpha_1 & \alpha_1 \\ 0 & \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 + \alpha_1 & \alpha_1 \\ 0 & \alpha_0 + \alpha_1 \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$