Model solutions for subject system identification of code MTCS102

Ans.1 a) The Probability of an Event can be explained by two of the most popular definitions are: i) the relative frequency definition, and ii) the classical definition.

1: The relative frequency definition

Suppose that a random experiment is repeated n times. If the event A occurs n_A times, then the probability of A, denoted by $P(A)$, is defined as $P(A) = \lim_{n \to \infty} \left(\frac{n_A}{n}\right)$

$\left(\frac{n_A}{n}\right)$ represents the fraction of occurrence of A in n trials.

For small values of n , it is likely that $\left(\frac{n_A}{n}\right)$ will fluctuate quite badly. But as n becomes larger and larger, we expect, $\left(\frac{n_A}{n}\right)$ to tend to a definite limiting value. For example, let the experiment be that of tossing a coin and A the event 'outcome of a toss is Head'. If n is the order of 100, $\left(\frac{n_A}{n}\right)$ may not deviate from ½ by more than, say ten percent and as n becomes larger and larger, we expect $\left(\frac{n_A}{n}\right)$ to converge to ½.

2: The classical definition:

The relative frequency definition given above has empirical flavor. In the classical approach, the probability of the event A is found without experimentation. This is done by counting the total number N of the possible outcomes of the experiment. If N_A of those outcomes are favorable to the occurrence of the event A, then $P(A) = \frac{N_A}{N}$ where it is assumed that all outcomes *are equally likely.*

probability
measure (to the various events on the sample space) to obey the following postulates or axioms:

P1) $P(A) \geq 0$

P2) $P(S) = 1$

P3) $(AB) = \varphi$, then $P(A + B) = P(A) + P(B)$

the symbol + is used to mean two different things;
namely, to denote the union of A and B and to denote the addition of two real numbers). Using Eq. 2.3, it is possible for us to derive some additional relationships:

i) If $AB \neq \varphi$, then $P(A + B) = P(A) + P(B) - P(AB)$

ii) Let $A_1, A_2, \ldots\ldots, A_n$ be random events such that:

i) a) $A_i A_j = \varphi$, for $i \neq j$ and

ii) b) $A_1 + A_2 + \ldots\ldots + A_n = S$. (2.5b)

iii) Then, ( ) ( ) ( ) ( ) $P A = P A A_1 + P A A_2 + \ldots\ldots + P A A_n$ (2.6)

iv) where A is any event on the sample space.

v) Note: $A_1, A_2, \cdots, A_n$ are said to be *mutually exclusive* (Eq. 2.5a) and *exhaustive*

vi) iii) $P(A) = 1 - P(A)$

Ans:2

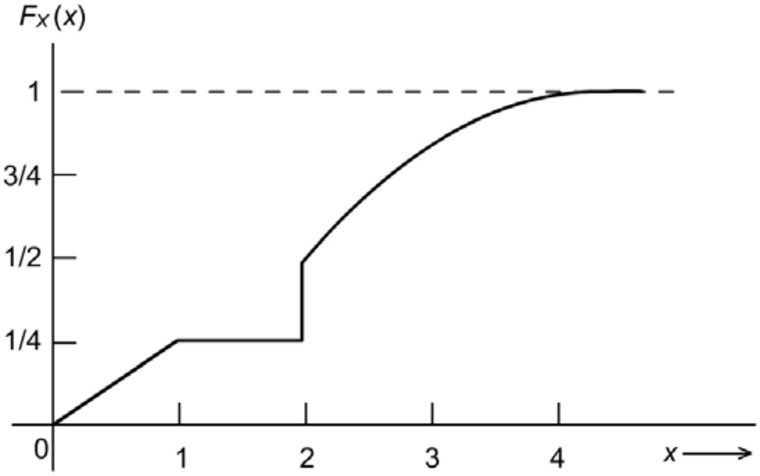
Probability Density Function (PDF), $f_X(x)$ is defined as the derivative of the Cumulative Derivative function F; that is

$f_X(x) = \frac{dF_X(x)}{dx}$ or $F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha$

The distribution function may fail to have a continuous derivative at a point $x = a$ for one of the two reasons:

i) the slope of the $f_X(x)$ is discontinuous at $x = a$

ii) $f_X(x)$ has a step discontinuity at $x = a$



As can be seen from the figure, $f_X(x)$ has a discontinuous slope at $x = 1$ and a step discontinuity at $x = 2$. In the first case, we resolve the ambiguity by taking f_X to be a derivative on the right.

The second case is taken care of by introducing the impulse in the probability domain. That is, if there is a discontinuity in F_X at $x = a$ of magnitude P_a, we include an impulse $P_a\, \delta(x - a)$ in the PDF $f_X(x) = \frac{1}{4}\delta\left(x + \frac{1}{2}\right) + \frac{1}{8}\delta\left(x - \frac{1}{2}\right) + \frac{1}{8}\delta\left(x - \frac{3}{2}\right) + \frac{1}{2}\delta\left(x - \frac{5}{2}\right)$ As in $f_X(x)$ has an impulse of weight 1/8 at $x = ½$ as $P\left[X = \frac{1}{2}\right] = \frac{1}{8}$ This impulse function cannot be taken as the limiting case of an even function (such as $\frac{1}{\varepsilon} ga\left(\frac{x}{\varepsilon}\right)$) because $\lim_{\varepsilon \to 0} \int_{½-\varepsilon}^{½} f_X(x)\, dx = \lim_{\varepsilon \to 0} \int_{½-\varepsilon}^{½} \frac{1}{8}\delta\left(x - \frac{1}{2}\right) dx \neq \frac{1}{16}$

However, $\lim_{\varepsilon \to 0} \int_{½-\varepsilon}^{½} f_X(x)\, dx = \frac{1}{8}$. This ensures

$$F_X(x) = \begin{cases} \frac{2}{8}, & -\frac{1}{2} \le x < \frac{1}{2} \\ \frac{3}{8}, & \frac{1}{2} \le x < \frac{3}{2} \end{cases}$$

Such an impulse is referred to as the *left-sided delta function.* As F_X is non-decreasing and $F_X(\infty) = 1$ we have

i) $f_X(x) \ge 0$

ii) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

Based on the behavior of CDF, a random variable can be classified as:

i) continuous (ii) discrete and (iii) mixed. If the CDF, $f_X(x)$ is a continuous function of x for all x , then X

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is a continuous random variable. If $f_X(x)$ is a staircase, then X corresponds to a discrete variable. We say that X is a mixed random variable if $f_X(x)$ is discontinuous but not a staircase.

Ans. 2 a).

The *mean value* (also called the *expected value, mathematical expectation* or simply *expectation*) of random variable X is defined as

$$m_X = E[X] = \overline{X} = \int_{-\infty}^{\infty} x\, f_X(x)\, d\, x$$

where E denotes the expectation operator. Note that m_X is a constant. Similarly, the expected value of a function of X, $g(X)$, is defined by

$$E[g(X)] = \overline{g(X)} = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, d\, x$$

For the special case of $g(X) = X^n$, we obtain the *n-th moment* of the probability distribution of the RV, X; that is,

$$E[X^n] = \int_{-\infty}^{\infty} x^n\, f_X(x)\, d\, x$$

The most widely used moments are the first moment ($n = 1$, which results in the mean value of Eq. 2.38) and the second moment ($n = 2$, resulting in the *mean square value* of X).

2. b)

Coming back to the central moments, we have the first central mome being always zero because,

$$E[(X - m_X)] = \int_{-\infty}^{\infty} (x - m_X)\, f_X(x)\, d\, x$$

$$= m_X - m_X = 0$$

Consider the second central moment

$$E\left[(X - m_X)^2\right] = E\left[X^2 - 2 m_X X + m_X^2\right]$$

From the linearity property of expectation,

$$E\left[X^2 - 2 m_X X + m_X^2\right] = E\left[X^2\right] - 2 m_X E[X] + m_X^2$$

$$= E\left[X^2\right] - 2 m_X^2 + m_X^2$$

$$= \overline{X^2} - m_X^2 = \overline{X^2} - \left[\overline{X}\right]^2$$

The second central moment of a random variable is called the ***variance*** and its (positive) square root is called the *standard deviation*. The symbol σ^2 is generally used to denote the variance. (If necessary, we use a subscript on σ^2)

The variance provides a measure of the variable's spread or randomness. Specifying the variance essentially constrains the effective width of the density function. This can be made more precise with the help of the *Chebyshev*

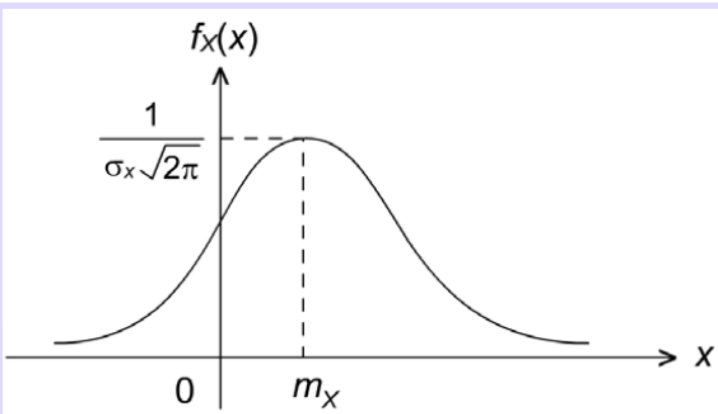
2.d)

iii) Gaussian

By far the most widely used PDF, in the context of communication theory is the Gaussian (also called *normal*) density, specified by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left[-\frac{(x - m_X)^2}{2\sigma_X^2}\right], \quad -\infty < x < \infty \tag{2.56}$$

where m_X is the mean value and σ_X^2 the variance. That is, the Gaussian PDF is completely specified by the two parameters, m_X and σ_X^2. We use the symbol $N(m_X, \sigma_X^2)$ to denote the Gaussian density[1]. In appendix A2.3, we show that $f_X(x)$ as given by Eq. 2.56 is a valid PDF.



Hence $F_X(m_X) = \int_{-\infty}^{m_X} f_X(x)\,dx = 0.5$

Consider $P[X \geq a]$. We have,

$$P[X \geq a] = \int_a^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left[-\frac{(x - m_X)^2}{2\sigma_X^2}\right] dx$$

This integral cannot be evaluated in closed form. By making a change of variable

$z = \left(\frac{x - m_X}{\sigma_X}\right)$, we have

$$P[X \geq a] = \int_{\frac{a - m_X}{\sigma_X}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$$= Q\left(\frac{a - m_X}{\sigma_X}\right)$$

where $Q(y) = \int_y^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \qquad (2.57)$

Note that the integrand on the RHS of Eq. 2.57 is $N(0, 1)$.