Mid Term Exam. Oct. 2017

Subject :- Structural Dynamics

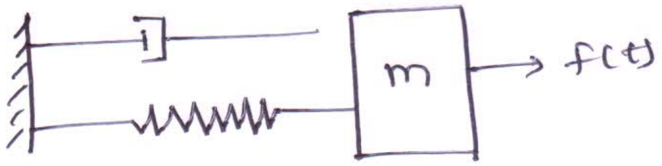
Subject code :- CVSE103

Q. No. 1 (a) Ans:-

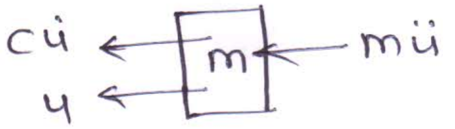
Derivation for Response of a damped SDOF system for free vibration.

i.e. $f(t) = 0$.

SDOF system model



SDOF system spring mass model



The equation of motion for damped free vibration

$$m\ddot{u} + c\dot{u} + ku = 0 \quad \text{--- (1)}$$

dividing by m to above eqn

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0$$

$$D^2 + \frac{c}{m}D + \frac{k}{m} = 0$$, Roots of the eqn are

$$D = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}}{2}$$

$$= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$\frac{c}{2m} = \eta\omega$$

$\therefore \quad c = 2m\eta\omega$

$\therefore \quad D = -\eta\omega \pm \omega\sqrt{\eta^2 - 1} = -\omega\eta \pm i\omega\sqrt{1-\eta^2}$

$\omega_d = \omega\sqrt{1-\eta^2}$

$\therefore$ The solution of above eqn ① is

$$\boxed{u(t) = C_1 e^{(-\omega\eta + i\omega_d)t} + C_2 e^{(-\eta\omega - i\omega_d)t}} \quad — ②$$

$$= e^{-\eta\omega t}[(C_1 \cos\omega_d t + C_1 \sin\omega_d t)] + [(C_2 \cos\omega_d t - i C_2 \sin\omega_d t)]$$

$\because \quad e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$$\therefore \quad u(t) = e^{-\eta\omega t}[A\cos\omega_d t + B\sin\omega_d t]$$

where, $A = C_1 + C_2$ & $B = C_1 - C_2$

The constants are calculated based on Boundry conditions.

$$u(t) = e^{-\eta\omega t}[A\cos\omega_d t + B\sin\omega_d t] \quad — ③$$

Applying Boundry conditions, At, $t = 0$, $u_0 = 0$ & $\dot{u}_0 = V_0$

$u_0 = A$, from eqn 3

$$\dot{u}(t) = V_0 = -\omega\eta[A\cos\omega_d t + B\sin\omega_d t]e^{-\eta\omega t} + e^{-\eta\omega t}[-A\omega_d \sin\omega_d t + B\omega_d \cos\omega_d t]$$

$\therefore \quad V_0 = -u_0\omega_n + B\omega_d \Rightarrow V_0 = -u_0\omega_n + B\omega_d$

$$\therefore \quad B = \frac{V_0 - u_0\omega_\eta}{\omega_d}$$

Sub. value of A & B in above eqn 3

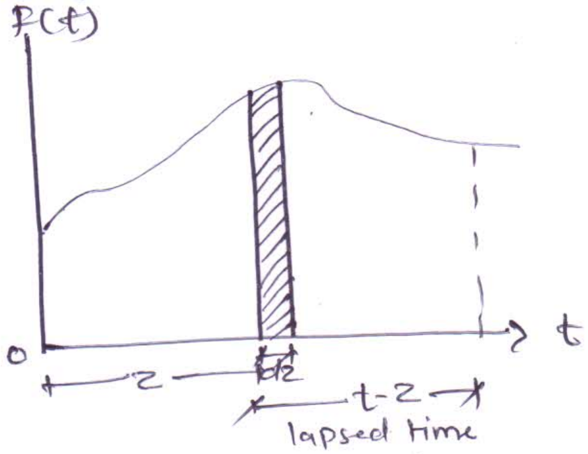
$$\boxed{u(t) = \left[u_0 \cos\omega_d t + \left(\frac{v_0 - u_0\omega_n}{\omega_d}\right)\sin\omega_d t\right] e^{-\eta\omega t}}$$

$u(t)$ = Total Response

u_0 = initial displacement

v_0 = initial velocity

Q. No. 1 (b) :- Ans :- Consider the ~~tmph~~ impulse of the force $F(z)$ as shown in fig. below at time t during the interval dz which represented by the shaded Area.



The impulse is equal to $F(z)\,dz$ and it is acting on a body of mass m, produces a change in velocity which can be determined Impulse loading from Newton's Second law of motion.

$$F = ma = m \cdot \frac{d\dot{u}}{dt}$$

After rearranging

$$d\dot{u} = \frac{F(z)\cdot(dz)}{m} \quad \text{—— (a)}$$

Here, $F(z)dz$ is impulse & $d\dot{u}$ is the change in velocity during the interval dz i.e. change in velocity during the interval dz.

This incremental velocity may be considered to be an initial velocity of the mass at time t.

This impulse $F(z)\,dz$ acting on the system represented by the undamped oscillator.

$$u(t) = u_c(t) + u_p(t) \quad \text{—} (1)$$

i.e. complementary function $u_c(t) = A\sin\omega_n t + B\cos\omega_n t$

$$u_c(t) = A\sin\omega_n t + B\cos\omega_n t \quad \&$$

particular integral $u_p(t)$ can be obtained by Duhamel's integral.

The change in velocity as initial velocity $\dot{u}_0$ together with initial displacement $u_0 = 0$ at time τ, producing a displacement at a later time t, is given by.

$$u_c(t) = A\sin\omega_n t + B\cos\omega_n t \quad \text{—} (2)$$

using initial condition.

$$t = 0, \quad u_0 = 0 \quad \& \quad \dot{u}_0 = d\dot{u} = V_0$$

sub. in eq^n (2) we get,

$$B = 0 \quad \& \quad A = \frac{d\dot{u}}{\omega_n} = \frac{V_0}{\omega_n}$$

$$\therefore \quad u(t) = \frac{V_0}{\omega_n}\sin\omega_n t \quad \text{—} (3)$$

But the duration (τ) represented is actually incremental time $(t-\tau)$

$$\therefore \quad u(t) = \frac{V_0}{\omega_n}\sin\omega_n(t-\tau)$$

$$= \frac{F(\tau)\,(d\tau)}{m\omega_n}\sin\omega_n(t-\tau) \quad \text{—} (4)$$

The loading may be represented as a series of short impulses at successive incremental times $d\tau$. Thus each short impulse producing its own response at time t of the form given in above eq^n (4). Therefore the total response created by all the incremental impulses is found by integrating eq^n (4) from, $t = 0$ to time t.

$$u(t) = \frac{1}{m\omega_n}\int_0^t F(z)\sin\omega_n(1-z)\cdot dz$$

This is ~~know~~ called as Duhomel's integral which represents the total displacement produced by the exciting force $F(z)$ acting on the undamped oscillator.

i) Total displacement by Duhamel's integral subjected to a loading i.e. $F(t) = F_o$, for undamped.

$$m\ddot{y} + ky = F_o$$

$$\ddot{y} + \frac{ky}{m} = F_o/m$$

$$\therefore\ u(t) = \frac{1}{\omega_n m}\int_0^t F_o \sin\omega_n(t-z)\cdot dz$$

$$= \frac{F_o}{\omega_n m}\int_0^t \sin\omega_n(t-z)\, dz$$

$$= \frac{F_o}{\omega_n m}\left[\frac{-\cos\omega_n(t-z)}{-\omega_n}\right]_0^t$$

$$= \frac{F_o}{\omega_n^2 m}\left[1-\cos\omega_n t\right]$$

$$\boxed{u(t) = \frac{F_o}{K}\left[1-\cos\omega_n t\right]}$$

ii) Total Displacement when $f(t) = F_o \cdot t$

$$\therefore\ u(t) = \frac{1}{\omega_n m}\int_0^t F_o\cdot t\sin\omega_n(t-z)\cdot dz$$

$$= \frac{F_o}{\omega_n m}\int_0^t z\sin\omega_n(t-z)\cdot dz$$

$$= \frac{F_o}{\omega_n m}\left\{\left[\frac{z\cos\omega_n(t-z)}{\omega_n}\right]_0^t - \int_0^t \frac{\cos\omega(t-z)}{\omega_n}dz\right\}$$