

Subject :- Structural Dynamics

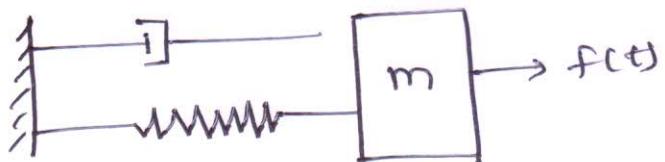
Subject code :- CVSE103

Q. NO. 1 (a) Ans:-

Derivation for response of a clamped  
SDOF system for free vibration.

i.e.  $f(t) = 0$ .

SDOF system model



SDOF system spring mass model



The equation of motion for clamped free vibration

$$m\ddot{u} + c\dot{u} + ku = 0 \quad \text{--- (1)}$$

dividing by  $m$  to above eqn

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0$$

$D^2 + \frac{c}{m}D + \frac{k}{m} = 0$ , Roots of the eqn are

$$D = -\frac{\frac{c}{m}}{2} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}$$

$$= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{4k}{m}}$$

$$\frac{c}{2m} = \gamma \omega$$

$$\therefore C = 2m\eta w$$

$$\therefore D = -\eta w \pm w\sqrt{\eta^2 - 1} = -w\eta \pm iw\sqrt{1-\eta^2}$$

$$\omega_d = w\sqrt{1-\eta^2}$$

$\therefore$  The solution of above eqn (1) is

$$y(t) = C_1 e^{(-wn+iwd)t} + C_2 e^{(-\eta w-iwd)t} \quad \text{--- (2)}$$

$$= e^{-\eta wt} [(C_1 \cos \omega_d t + C_1 \sin \omega_d t)] + \\ [C_2 \cos \omega_d t - i C_2 \sin \omega_d t]$$

$$\therefore e^{i\phi} = \cos \phi + i \sin \phi$$

$$e^{-i\phi} = \cos \phi - i \sin \phi$$

$$\therefore y(t) = e^{-\eta wt} [A \cos \omega_d t + B \sin \omega_d t]$$

$$\text{where, } A = C_1 + C_2 \text{ & } B = C_1 - C_2$$

The constants are calculated based on  
Boundary conditions.

$$u(t) = e^{-\eta wt} [A \cos \omega_d t + B \sin \omega_d t] \quad \text{--- (3)}$$

Applying Boundary conditions, At,  $t=0$ ,  $y_0=0$  &  $y_0=v_0$

$$\therefore y_0 = A, \text{ from eqn 3}$$

$$y(t) = v_0 = -w\eta [A \cos \omega_d t + B \sin \omega_d t] e^{-\eta wt} \\ + e^{-\eta wt} [-A \omega_d \sin \omega_d t + B \omega_d \cos \omega_d t]$$

$$\therefore v_0 = -y_0 w\eta + B \omega_d \Rightarrow v_0 = -y_0 w\eta + B \omega_d$$

$$\therefore B = \frac{v_0 - y_0 w\eta}{\omega_d}$$

Sub. value to A & B in above eqn 3

$$u(t) = [u_0 \cos \omega_d t + \left( \frac{v_0 - u_0 \omega_d}{\omega_d} \right) \sin \omega_d t] e^{-\eta \omega_d t}$$

$u(t)$  : Total Response

$u_0$  = initial displacement

$v_0$  = initial velocity.

Q. No. 1 (b) :- Ans:- Consider the total impulse of the force  $F(z)$  as shown in fig. below at time  $t$  during the interval  $dz$   $F(t)$

which represented by the shaded Area.

The impulse is equal

to  $F(z)dz$  and it is acting on a body of mass  $m$ , produces a change in velocity which can be determined Impulse, loading

from Newton's Second law of motion.

$$F = ma = m \cdot \frac{du}{dt}$$

After rearranging

$$du = \frac{F(z) \cdot (dz)}{m} \quad \text{--- (a)}$$

Here,  $F(z)dz$  is impulse &  $du$  is the change in velocity during the interval  $dz$  i.e. change in velocity during the interval  $dz$ .

This incremental velocity may be considered to be an initial velocity of the mass at time  $t$ .

This impulse  $F(z)dz$  acting on the system represented by the undamped oscillator.

$$u(t) = u_c(t) + u_p(t) \quad \text{--- (1)}$$

i.e. complementary function  $u_c(t) = A \sin \omega_n t + B \cos \omega_n t$

$$u_c(t) = A \sin \omega_n t + B \cos \omega_n t \quad \&$$

particular integral  $u_p(t)$  can be obtained by Duhamel's integral.

The change in velocity as initial velocity  $v_0$  together with initial displacement  $u_0 = 0$  at time  $z$ , producing a displacement at a later time  $t$ , is given by.

$$u_c(t) = A \sin \omega_n t + B \cos \omega_n t \quad \text{--- (2)}$$

using initial condition,

$$t = 0, u_0 = 0 \quad \& \quad v_0 = \frac{du}{dt} = v_0$$

sub. in eqn (2) we get

$$B = 0 \quad \& \quad A = \frac{v_0}{\omega_n} = \frac{v_0}{\omega_n}$$

$$\therefore u(t) = \frac{v_0}{\omega_n} \sin \omega_n t \quad \text{--- (3)}$$

But the duration (2) represented is actually incremental time  $(t-z)$

$$\therefore u(t) = \frac{v_0}{\omega_n} \sin \omega_n (t-z)$$

$$= \frac{F(z)(dz)}{m \omega_n} \sin \omega_n (t-z) \quad \text{--- (4)}$$

The loading may be represented as a series of short impulses at successive incremental times  $dz$ . Thus each short impulse producing its own response at time  $t$  of the form given in above eqn (4). Therefore the total response created by all the incremental impulses is found by integrating eqn (4) from  $t=0$  to time  $t$ .

$$u(t) = \frac{1}{m\omega_n} \int_0^t F(z) \sin \omega_n(t-z) dz$$

This is known called as Duhamel's integral which represents the total displacement produced by the exciting force  $F(z)$  acting on the undamped oscillator.

- i) Total displacement by Duhamel's integral subjected to a loadig, i.e.  $F(t) = F_0$ , for undamped.

$$m\ddot{u} + k u = F_0$$

$$\ddot{u} + \frac{k u}{m} = F_0/m$$

$$\therefore u(t) = \frac{1}{\omega_n m} \int_0^t F_0 \sin \omega_n(t-z) dz$$

$$= \frac{F_0}{\omega_n m} \int_0^t \sin \omega_n(t-z) dz$$

$$= \frac{F_0}{\omega_n m} \left[ \frac{-\cos \omega_n(t-z)}{-\omega_n} \right]_0^t$$

$$= \frac{F_0}{\omega_n^2 m} [1 - \cos \omega_n t]$$

$$u(t) = \frac{F_0}{k} [1 - \cos \omega_n t]$$

- ii) Total Displacement, when  $F(t) = F_0 \cdot t$

$$\therefore u(t) = \frac{1}{\omega_n m} \int_0^t F_0 \cdot z \sin \omega_n(t-z) dz$$

$$= \frac{F_0}{\omega_n m} \int_0^t z \sin \omega_n(t-z) dz$$

$$= \cancel{\frac{F_0}{\omega_n m}} \frac{F_0}{\omega_n m} \left\{ \left[ \frac{z \cos \omega_n(t-z)}{\omega_n} \right]_0^t \right.$$

$$\left. - \int_0^t \frac{\cos \omega_n(t-z)}{\omega_n} dz \right\}$$

$$= \frac{F_0}{\omega_{nm}} \left\{ \left[ \frac{t}{\omega_n} \right] - \left[ \frac{\sin \omega_n (t-2)}{-\omega_n^2} \right]^t \right\}$$

$$\therefore u(t) = \frac{F_0}{\omega_{nm}} \left\{ \frac{t}{\omega_n} - \frac{\sin \omega_n t}{\omega_n^2} \right\}$$

$$u(t) = \frac{F_0}{k_s} \left\{ t - \frac{\sin \omega_n t}{\omega_n} \right\}$$

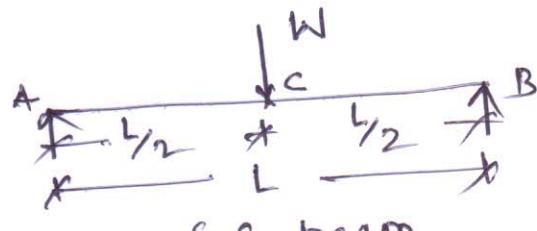
Q. NO. 2. (b) soln:-

We know,

$$T = \frac{2\pi}{\omega_n}$$

where,  $T$ : Natural Time period

$\omega_n$ : Natural frequency



Deflected shape

$m$  = mass of structure.

$k$  = stiffness of structure.

But,  $k$  = stiffness for simply supported beam subjected to central load,

$$\text{i.e. } k = \frac{W}{\Delta}, \text{ where,}$$

$$k = \frac{W}{\Delta}$$

$$= \frac{48EI \cdot W}{WL^3}$$

$$k = \frac{48EI}{L^3}$$

$$\therefore \omega_n = \sqrt{\frac{48EI}{m \cdot L^3}}$$

$$T = \frac{2\pi}{\omega_n}$$

$\Delta$ : deflection at centre of S.S. beam

$$\text{i.e. } \Delta = \frac{WL^3}{48EI}$$

$W$  = Load. at centre