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Mid Semester Examination – October - 2017

Branch: M.Tech. (Civil Engineering with Specialisation in Structural Engineering) Sem.:- I
Subject with Subject Code: CVSE101: Theory of Elasticity & Plasticity Marks: 20
Date: - 09/10/2017 Time: - 1 Hr.

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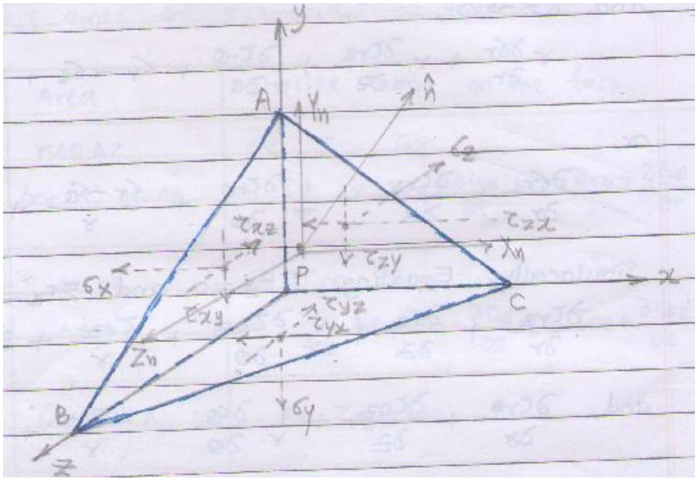
SOLUTION

Q. No.1 Attempt any one of the following (08)

a) Derive Cauchy's formula for stress components on an arbitrary plane. Hence find Characteristic Equation of Principal Stresses.

Solution:



If stress components acting on three mutually perpendicular planes passing through a point are known then stress components acting on any arbitrary plane passing through that point can be obtained. Let, the three mutually perpendicular planes be x, y, z planes and let the arbitrary plane be identified by its outward normal $\hat{n}$ whose direction cosines are l, m, n. Consider a small tetrahedron at P with three of its faces normal to the coordinate axis and the inclined face having its normal parallel to $\hat{n}$. Let h = perpendicular distance from P to inclined face. Let, R_n is the resultant stress vectors on face ABC. This can be resolved into components X_n, Y_n, Z_n parallel to the three axis X, Y and Z. if, A = area of inclined face ABC.

Area APB = Projection of ABC on YZ plane = A.l

Area BPC = Projection of ABC on XZplane = A.m

Area APC = Projection of ABC on XYplane = A.n

Let body force components in x, y, z directions be $B_x, B_y..and, B_z$ per unit volume. The volume of tetrahedron is $\frac{1}{3}Ah$. Therefore considering equilibrium in x-direction,

$$X_n A - \pm_x Al - {}^2{}_{yx}.Am - {}^2{}_{zx}An + \frac{1}{3}AhB_x = 0$$

Cancelling, A and as h → 0: plane ABC → point P,

Therefore,

$$X_n = \pm_{xx}.l + {}^2{}_{yx}.m + {}^2{}_{zx}.n;$$

Similarly considering $\sum F_y = 0$ and $\sum F_z = 0$;

$$Y_n = 2_{xy} l + \pm_{yy}.m + 2_{yz}.n \text{ , and } Z_n = 2_{xz}.l + 2_{yz}.m + \pm_{zz}.n$$

These three equations are known as surface conditions or Cauchy's conditions.

$$\begin{Bmatrix} X_n \\ Y_n \\ Z_n \end{Bmatrix} = [\pm]^T \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$$

The planes on which stresses are wholly normal are called principle stresses. For planes on which Rn is normal; $X_n = \pm.l, Y_n = \pm.m, Z_n = \pm.n$ where $\pm$ is magnitude of normal resultant.

From Cauchy's formula,

$$\begin{Bmatrix} X_n \\ Y_n \\ Z_n \end{Bmatrix} = \begin{Bmatrix} \pm.l \\ \pm.m \\ \pm.n \end{Bmatrix} = \begin{bmatrix} \pm_{xx} & 2_{xy} & 2_{xz} \\ 2_{yx} & \pm_{yy} & 2_{yz} \\ 2_{zx} & 2_{zy} & \pm_{zz} \end{bmatrix} \times \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$$

$$\therefore \begin{bmatrix} \pm_{xx} - \pm & 2_{xy} & 2_{xz} \\ 2_{yx} & \pm_{yy} - \pm & 2_{yz} \\ 2_{zx} & 2_{zy} & \pm_{zz} - \pm \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

This is set of simultaneous equations. Hence its determinant must be zero.

$$\therefore \begin{vmatrix} \pm_{xx} - \pm & 2_{xy} & 2_{xz} \\ 2_{yx} & \pm_{yy} - \pm & 2_{yz} \\ 2_{zx} & 2_{zy} & \pm_{zz} - \pm \end{vmatrix} = 0 \text{ ; } \therefore \pm^3 - I_1 \pm^2 + I_2 \pm - I_3 = 0$$

This equation is known as characteristic equation and roots of this equation are called principle stresses.

b) The state of stress characterized by $[\pm_{ij}]$ is given below. Resolve the given state into Hydrostatic state and Pure Shear state. Determine the normal and shearing stress on octahedral plane.

$$[\pm_{ij}] = \begin{bmatrix} 10 & 4 & 7 \\ 4 & 8 & 2 \\ 7 & 2 & 12 \end{bmatrix} MPa$$

Solution:

State of Hydrostatic state of is given by, $\pm_h = \frac{1}{3} I_1 = (\pm_x + \pm_y + \pm_z) = \frac{1}{3}(10+8+12) = 10MPa$

Therefore,

$$[\pm_h] = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} MPa$$

and, Pure shear state is, $[\pm_d] = \begin{bmatrix} 0 & 4 & 7 \\ 4 & 2 & 2 \\ 7 & 2 & -2 \end{bmatrix} MPa$

Stress Invariants are given by,

First Invariant, $I_1 = \pm_x + \pm_y + \pm_z = (10+8+12) = 30$ and,

Second Invariant, $I_2 = \begin{vmatrix} \pm_x & 2_{xy} \\ 2_{yx} & \pm_y \end{vmatrix} + \begin{vmatrix} \pm_y & 2_{yz} \\ 2_{zy} & \pm_z \end{vmatrix} + \begin{vmatrix} \pm_x & 2_{xz} \\ 2_{zx} & \pm_z \end{vmatrix} = \begin{vmatrix} 10 & 4 \\ 4 & 8 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 12 \end{vmatrix} + \begin{vmatrix} 10 & 7 \\ 7 & 12 \end{vmatrix} = 64 + 92 + 71 = 227$

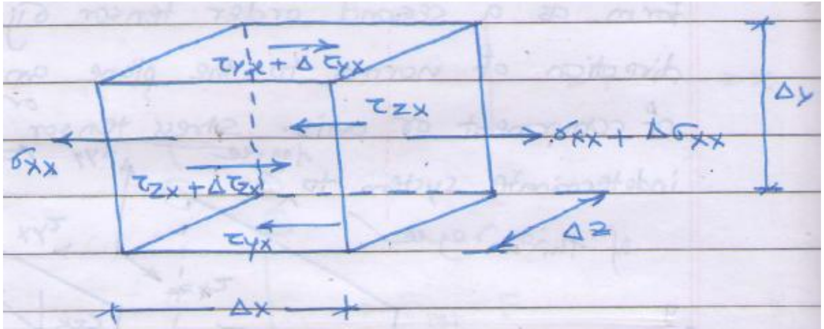
Therefore the octahedral normal stress is $\sigma_{oct} = \frac{1}{3} x I_1 = \frac{1}{3} x 30 = 10 MPa$.

The octahedral normal stress is $\tau_{oct} = \frac{\sqrt{2}}{3} x (I_1^2 - 3I_2)^{\frac{1}{2}} = \frac{\sqrt{2}}{3} x (30^2 - (3x227))^{\frac{1}{2}} = 51.62 MPa$

Q. No. 2 Attempt any three of the following: (12)

a) Derive governing differential equation of equilibrium for 3-D state of stress in Cartesian coordinate system.

Solution:



Consider, elementary volume $\Delta X \times \Delta Y \times \Delta Z$, subjected to a state of stress and body forces as shown. Forces on elementary volume acting in X-direction only are shown in figure. Let, body force vector be $\vec{B} = B_x.\hat{i} + B_y.\hat{j} + B_z.\hat{k}$.

Thus, for static case, considering $\Sigma F_x = 0$,

$$(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial_x}\Delta x - \sigma_{xx})\Delta y.\Delta z + (\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y - \tau_{yx})\Delta x.\Delta z + (\tau_{zx} + \frac{\partial \tau_{zx}}{\partial_z}\Delta x - \tau_{zx})\Delta x.\Delta y + B_x.\Delta x.\Delta y.\Delta z = 0$$

$$\therefore \frac{\partial \sigma_{xx}}{\partial_x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial_z} + B_x = 0$$

Similarly, considering $\Sigma F_y = 0$,

$$\therefore \frac{\partial \tau_{xy}}{\partial_x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial_z} + B_y = 0$$

And considering $\Sigma F_z = 0$,

$$\therefore \frac{\partial \tau_{xz}}{\partial_x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial_z} + B_z = 0$$

b) What do you understand by Airy's Stress Function? Explain how to obtain constant stress field and strain field with linearly varying displacement field.

Solution:

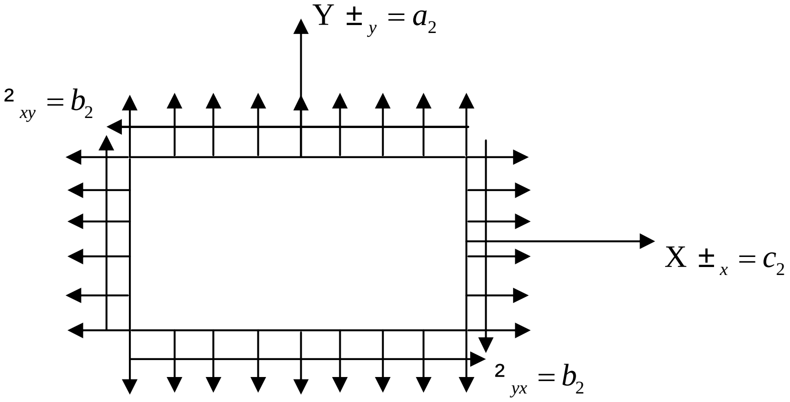
The scalar function $\phi(x, y)$ be such that, $\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + \Psi; \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + \Psi; \tau_{xy} = -\frac{\partial^2 \phi}{\partial xy}$, where $\phi = \phi(x, y)$; is an arbitrary form called the Airy stress function. In absence of body forces, $\cdot = 0$; and $\nabla \phi = 0$. Thus plane problem of elasticity gets reduced to single equation in terms of ϕ. The Airy stress function is also independent of elastic constants in the absence of body forces. Thus, the stress field flon plane stress and plane strain will be identical and independent of elastic constants. However, a resulting strain and displacement calculated from these stress fields will be different as Hooke's law and stran displacement relations are different.

In Cartesian coordinate system, generally Airy stress function will take a form as,

$$\phi(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}a_{ij}x^i y^j$$

To write a stress function of particular order, Pascal's triangle is to be used.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & x & & y & & \\ & x^2 & & xy & & y^2 & \\ x^3 & & x^2y & & xy^2 & & y^3 \\ -- & -- & -- & -- & -- & -- & -- \end{array}$$



With second degree polynomial, with $m = 2, and, n = 2$,

$$Let, \phi(x,y)=\frac{a_2x^2}{2}+b_2.x.y+\frac{c_2y^2}{2}\ldots thus,$$

$$\sigma_{xx}=\frac{\partial^2\phi}{\partial y^2}=c_2\ldots,\sigma_{yy}=\frac{\partial^2\phi}{\partial x^2}=a_2\ldots,\tau_{xy}=-\frac{\partial^2\phi}{\partial xy}=-b_2$$

Thus, in absence of body force or constant body force $\Psi = 0$; second order polynomial gives constant stress field and constant strain field. It gives a linearly varying displacement field.

c) Explain constitutive relation for linearly elastic, isotropic material in State of Plane Strain.
Solution:

Let condition of plane strain exists in X-Y plane. From constitutive law for Three Dimensional state of stress,

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix}=\frac{E}{(1-2\nu)(1+\nu)}\begin{bmatrix}(1-\nu) & \nu & \nu & 0 & 0 & 0\\ \nu & (1-\nu) & \nu & 0 & 0 & 0\\ \nu & \nu & (1-\nu) & 0 & 0 & 0\\ 0 & 0 & 0 & (1-2\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & (1-2\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{Bmatrix}$$

For plane strain condition, $\varepsilon_{xy},\varepsilon_{xx}\ldots and\ldots\varepsilon_{yy}\ldots exist\ldots and\ldots\varepsilon_{zz}=\varepsilon_{yz}=\varepsilon_{zx}=0$. Thus the above constitutive law reduces to,

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}=\frac{E}{(1-2\nu)(1+\nu)}\begin{bmatrix}(1-\nu) & \nu & 0\\ \nu & (1-\nu) & 0\\ 0 & 0 & (1-\nu)\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{Bmatrix}$$

But, $\pm_{zz} \neq 0, \therefore \pm_{zz} = \dfrac{{}^{3}E}{(1-2{}^{3})(1+{}^{3})}(\pounds_{xx} + \pounds_{yy})$

d) For a point in a linearly elastic isotropic body, the deviatoric strain tensor is given by,

$$\left[\pounds_d\right] = \begin{bmatrix} -9 & 8 & 9 \\ 8 & 16 & 2 \\ 9 & 2 & -9 \end{bmatrix} x10^{-4}$$

The volumetric strain is 16 x 10^{-4} and Lame's constants are $^{a} = 120GPa, \text{«} = 80GPa$ **. Find state of stress at the point.**

Solution:

The relation between $\left[\pm_d\right] \ldots and \left[\pounds_d\right]$ is given by,

$$\left[\pm_d\right] = 2 \times \text{«}\left[\pounds_d\right]$$

$$\therefore \left[\pm_d\right] = 2 \times 80 \times 10^3 \times \begin{bmatrix} -9 & 8 & 9 \\ 8 & 16 & 2 \\ 9 & 2 & -9 \end{bmatrix} x10^{-4}$$

$$\therefore \left[\pm_d\right] = \begin{bmatrix} -144 & 128 & 144 \\ 128 & 256 & 32 \\ 144 & 32 & -144 \end{bmatrix} MPa$$

Given that, $e = \pounds_{xx} + \pounds_{yy} + \pounds_{zz} = 16 \times 10^{-4}$, therefore, $\pounds_m = \dfrac{e}{3} = \dfrac{16 \times 10^{-4}}{3} = 5.33 \times 10^{-4}$

$$\therefore \left[\pounds\right] = \begin{bmatrix} -3.67 & 8 & 9 \\ 8 & 21.33 & 2 \\ 9 & 2 & -3.67 \end{bmatrix} \times 10^{-4}$$

The state of stress is given by,

$\pm_{ij} = 2\text{«}\pounds_{ij} + {}^{a}\partial_{ij}\pounds_{kk}$

$\pm_{xx} = (2 \times 80 \times 10^3 \times -3.67 \times 10^{-4}) + (120 \times 10^3 \times 16 \times 10^{-4}) = 133.28MPa$

$\pm_{yy} = (2 \times 80 \times 10^3 \times 21.33 \times 10^{-4}) + (120 \times 10^3 \times 16 \times 10^{-4}) = 533.28MPa$

$\pm_{zz} = (2 \times 80 \times 10^3 \times -3.67 \times 10^{-4}) + (120 \times 10^3 \times 16 \times 10^{-4}) = 133.28MPa$

${}^{2}{}_{xy} = {}^{2}{}_{xy} = 2\text{«}\pounds_{xy} = 2 \times 80 \times 10^3 \times 8 \times 10^{-4} = 128MPa$

${}^{2}{}_{xz} = {}^{2}{}_{zx} = 2\text{«}\pounds_{xy} = 2 \times 80 \times 10^3 \times 9 \times 10^{-4} = 144MPa$

${}^{2}{}_{yz} = {}^{2}{}_{zy} = 2\text{«}\pounds_{xy} = 2 \times 80 \times 10^3 \times 2 \times 10^{-4} = 32MPa$

$$\therefore \left[\pm\right] = \begin{bmatrix} 133.28 & 128 & 144 \\ 128 & 533.28 & 32 \\ 144 & 32 & 133.28 \end{bmatrix} MPa$$